PART VIII

Results based on the NS equations in fully developed turbulence

1. The turbulence fact : Definition, observations and universal features of turbulence
2. The governing equations
3. Statistical description of turbulence
4. Turbulence modeling
5. Turbulent wall bounded flows
6. Homogeneous Isotropic Turbulence
7. Homogeneous Shear Flows
8. Results based on the equations of the dynamics in fully developed turbulence

8. Results based on the equations of the dynamics in fully developed turbulence
   - Tensorial general expressions
   - von Kármán equation
   - Kolmogorov 4/5 law
   - Bibliography
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Dynamics based results in fully developed turbulence

Tensorial general expressions

Considering homogeneous isotropic incompressible turbulence.

$$
\begin{aligned}
Q_i(\mathbf{r}) &= A(r)r_i \underset{\text{ex.}}{=} \overline{p(\mathbf{x})u_i(\mathbf{x}+\mathbf{r})} \\
R_{ij}(\mathbf{r}) &= F(r)r_ir_j + G(r)\delta_{ij} \underset{\text{ex.}}{=} \overline{u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})} \\
S_{ij\ell}(\mathbf{r}) &= A(r)r_ir_jr_\ell + B(r)(r_i\delta_{j\ell} + r_j\delta_{i\ell}) + D(r)r_\ell\delta_{ij} \\
&\underset{\text{ex.}}{=} \overline{u_i(\mathbf{x})u_j(\mathbf{x})u_\ell(\mathbf{x}+\mathbf{r})}
\end{aligned}
$$

where A, F, G, B, D are arbitrary scalar functions of r^2 ; all even function of r (by isotropy)

- $R_{ij}(\mathbf{r})$ symmetric in i, j : $R_{ij}(\mathbf{r}) = R_{ji}(\mathbf{r})$
- $S_{ijl}(\mathbf{r})$ symmetric in i, j : $S_{ijl}(\mathbf{r}) = S_{ji\ell}(\mathbf{r})$

General form for HIT

Expression of Q_i

- Continuity condition : $\partial_i u_i = 0$

$$\frac{\partial Q_i(\mathbf{r})}{\partial r_i} = 3A + r\frac{\partial A}{\partial r} = 0 \;, \forall \mathbf{r}$$

$\Longrightarrow A(r) = 0$ assuming regularity at $r = 0$.

- First order tensor

$$Q_i(\mathbf{r}) \equiv 0 \;, \quad \forall \mathbf{r}.$$

Tensorial general form

Expression of R_{ij}

- Continuity condition :

$$\frac{\partial R_{ij}(\mathbf{r})}{\partial r_i} = r_j\left(4F + r\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial G}{\partial r}\right) = 0 \;, \forall \mathbf{r}$$

$$\Longrightarrow 4F + r\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial G}{\partial r} = 0$$

Tensorial general expression

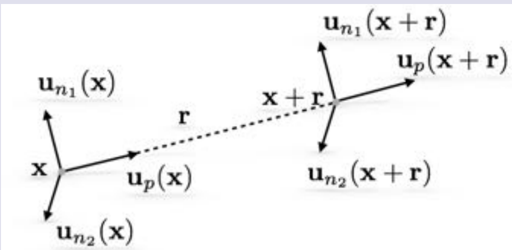
Expression of R_{ij}

- Longitunal and lateral velocity correlations

$$\overline{u_p(\mathbf{x})u_p(\mathbf{x}+\mathbf{r})} \equiv u^2 f(r)$$

$$\overline{u_n(\mathbf{x})u_n(\mathbf{x}+\mathbf{r})} \equiv u^2 g(r)$$

  - $f(r)$, $g(r)$: even scalar functions
  - u_{rms} : $u^2 \equiv \overline{u_p^2} = \overline{u_n^2} = \frac{1}{3}\overline{u_i^2}$



General form for HIT

Expression of $S_{ij\ell}$

- From the general form, one obtains

$$s_{ij\ell}(\mathbf{r}) = u^3\left[\frac{k - rk'}{2r^3}r_i r_j r_\ell + \frac{2k + rk'}{4r}\left(r_i\delta_{j\ell} + r_j\delta_{i\ell}\right) - \frac{k}{2r}r_\ell\delta_{ij}\right] \;,$$

where $k(r)$ is the single scalar function determining the triple velocity correlation and

$$k'(r) \equiv \frac{\partial k(r)}{\partial r} \;.$$

- Note that

$$S_{iji}(\mathbf{r}) = \frac{1}{2}u^3\left[k' + \frac{4}{r}k\right]r_j \equiv \frac{1}{2}K(r)r_j$$

VKH equation

- Write NS equation at points $\mathbf{x}$ and $\mathbf{x}' = \mathbf{x} + \mathbf{r}$.
- Summing and averaging with $u_i \equiv u_i(\mathbf{x}, t)$ and $u'_i \equiv u_i(\mathbf{x} + \mathbf{r}, t)$, in order to write

$$\partial_t \overline{u_i u'_j} = \ldots$$

VKH equation

- Previous equation leads to

$$\begin{aligned}\frac{\partial R_{ij}(\mathbf{r})}{\partial t} &= \frac{\partial}{\partial r}\left[\overline{u_i u_k u'_j} - \overline{u_i u'_k u'_j}\right] \\ &+ \left[\frac{\partial}{\partial r_i}\overline{p u'_j} - \frac{\partial}{\partial r_j}\overline{p' u_i}\right] \\ &+ 2\nu \frac{\partial^2 R_{ij}(\mathbf{r})}{\partial r_\ell \partial r_\ell}\end{aligned}$$

VKH equation

- Putting $i = j$

$$\frac{\partial R(r)}{\partial t} = \frac{1}{2}\left(r\frac{\partial}{\partial r} + 3\right) K(r) + 2\nu \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) R(r)$$

- First integral of this equation gives

$$\frac{\partial (u^2 f)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right)(u^3 k) + 2\nu \left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)(u^2 f)$$

Structure functions

Considering the velocity structure functions

- 2nd order

$$B_{ik}(\mathbf{r}) = \overline{\delta u_i(\mathbf{r}) \delta u_k(\mathbf{r})} = \overline{(u_i(\mathbf{x}+\mathbf{r}) - u_i(\mathbf{x}))(u_k(\mathbf{x}+\mathbf{r}) - u_k(\mathbf{x}))}$$

- 3rd order

$$\begin{aligned}B_{ik\ell}(\mathbf{r}) &= \overline{\delta u_i(\mathbf{r}) \delta u_k(\mathbf{r}) \delta u_\ell(\mathbf{r})} \\ &= \overline{(u_i(\mathbf{x}+\mathbf{r}) - u_i(\mathbf{x}))(u_k(\mathbf{x}+\mathbf{r}) - u_k(\mathbf{x}))(u_\ell(\mathbf{x}+\mathbf{r}) - u_\ell(\mathbf{x}))}\end{aligned}$$

And replacing in the von Kármán Howarth equation yields

$$-\frac{2}{3}\varepsilon + \frac{1}{2}\frac{\partial B_{pp}}{\partial t} = \frac{1}{6r^4}\frac{\partial}{\partial r}(r^4 B_{ppp}) - \frac{\nu}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial}{\partial r} B_{pp}\right)$$

with

$$\frac{\partial E}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\overline{u_i^2} = -\varepsilon = \frac{3}{2}\frac{\partial}{\partial t}\overline{u_p^2}\,.$$

Kolmogorov 4/5 law

VKH to 4/5 law

- Neglecting the time derivative term compared to ε .
- Neglecting the dissipative term in the inertial range
- Integrating over r
- One obtains in the inertial Range

$$B_{ppp} = \overline{(\delta u_p(\mathbf{r}))^3} = -\frac{4}{5}\varepsilon r$$

- But observed only for very large Reynolds number!!!

Bibliography

Books

- Batchelor 1953, The Theory of Homogeneous Turbulence
- Frisch 1995, Turbulence
- Pope 2000, Turbulent Flow
- Monin & Yaglom, Statistical Fluids Mechanics, Mechanics of Turbulence

Final Exam

Objective

- Start by determining the relation, equivalent to the '4/5 law', for the burgers equations which is written for $\langle \delta u^3 \rangle$. A rigorous demonstration is expected.
- Extend this demonstration to the case of 3D turbulent flow by writing the von Karman equation and to the famous Kolmogorov 4/5 law.
- Your mission consists in giving all the details and assumptions of the derivation.
- Write the demonstration for the equivalent law in the case of the passive scalar dynamics.