PART VII
Homogeneous Shear Flows

Definition and observations

Craya's definition of the homogeneity
- Isotropy broken by the existence of non uniform mean velocity field \( \bar{u} \).
- Homogeneity is preserved, Craya in 1958 : the mean velocity gradient matrix \( \bar{A} \):

\[
\bar{A}(t) \equiv \nabla \bar{u}(t)
\]  \hspace{1cm} (43)

is uniform in space, but can eventually be a time-varying quantity.
- Incompressibility constrain applied to \( \bar{u} \) \( \Rightarrow \) \( A_{ii} = 0 \).
- The mean velocity field is then given by

\[
\bar{u}(x, t) = \bar{A}(t)x + \bar{u}(0, t)
\]  \hspace{1cm} (44)
Definition and observations (cont’d)

Craya’s definition of the homogeneity

- The mean velocity field is then given by
  \[ \bar{u}(x,t) = A(t)x + u(0,t) \] (45)

- \textit{Rmq:} diverge when \( |x| \to +\infty \to \text{Not physical!} \)

- Indeed no characteristic length-scales for \( \bar{u} \), only a time-scale usually estimated as \( 1/\|A\| \).

- The homogeneity constrain implies that all the terms of the budget equation, appearing as turbulent spatial fluxes, are zero.

- The main difference with the isotropic case is that the production terms (of Reynolds stresses, kinetic energy, turbulent scalar flux, scalar variance, ...) are non zero.

- The field \( \bar{u} \) being anisotropic, the turbulence production will be also (at all the scales affected by the production phenomena), and this anisotropy will propagate via the non-linear interactions (turbulent cascade).

- Identify the effects directly driven by the production and those driven by the non-linear mechanisms of the cascade.

- All the matrices are not admissible to define an mean velocity gradient. Inserting the velocity expression from (45) in the mean momentum equation, we obtain the following compatibility condition for \( A \)

  \[ \left( \frac{d}{dt}A + A^2 \right) \text{ is symmetric} \] (46)

- Hereafter, we consider the constant pure shear

  \[ A = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \] (47)

  where \( S = d\bar{u}_1/dx_2 \) is the shear rate.

- Non uniform mean field \( \implies \)
  - Fast terms, which explicitly include \( \bar{u} \).
  - Slow terms, which contain only the fluctuating field \( u' \).

- The fast terms are \textit{instantaneously} modified if \( \bar{u} \) (so here \( A \)) is perturbed.

- The slow terms are modified with a \textit{delay}, associated to the characteristic timescale of the non linear mechanisms which propagate this perturbation.

- Momentum equation (in absence of external force) becomes

  \[ \frac{\partial u_i'}{\partial t} + A_{jk}x^k \frac{\partial u'_j}{\partial x_j} + A_{ij}u'_j + \frac{\partial}{\partial x_i}(u'_iu'_j) = - \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u'_i}{\partial x_k \partial x_k} \] (48)

  \[ \equiv \frac{D}{Dt} u'_i \]
Slow and fast terms

Craya’s definition of the homogeneity

- Momentum equation (in absence of external force)

\[
\frac{\partial u_i'}{\partial t} + A_{jk} x_k \frac{\partial u_j'}{\partial x_j} + A_{ij} u_j' + \frac{\partial}{\partial x_j} (u_i' u_j') = - \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_k \partial x_k} \tag{49}
\]

- The only slow term based on the velocity is the non linear term \(\frac{\partial}{\partial x_j} (u_i' u_j')\).

- All the fast terms are linear in \(u'\).

Slow and fast Pressure terms

Craya’s definition of the homogeneity

- Pressure terms given by Poisson equation

\[- \Delta p' = 2 A_{lm} \frac{\partial u_m'}{\partial x_l} + \frac{\partial u_m'}{\partial x_l} \frac{\partial u_l'}{\partial x_m} \tag{50}\]

- By linearity, the pressure \(p'\) is decomposed into: a fast \(p'_f\) and a fast \(p'_s\) field

\[- \Delta p'_f = 2 A_{lm} \frac{\partial u_m'}{\partial x_l}, \quad - \Delta p'_s = \frac{\partial u_m'}{\partial x_l} \frac{\partial u_l'}{\partial x_m} \tag{51}\]

- Then, eq. (49) can be rewritten as

\[
\frac{\partial u_i'}{\partial t} + A_{jk} x_k \frac{\partial u_j'}{\partial x_j} + A_{ij} u_j' + \frac{\partial}{\partial x_j} (u_i' u_j') = - \frac{\partial p_f'}{\partial x_i} - \frac{\partial p_s'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_k \partial x_k} \tag{52}
\]

Slow and fast Pressure terms

Time scales

- Characteristic Fast time scale based on \(\bar{u}\) :

\[\tau_f \sim S^{-1}\]

- Characteristic Slow time scale based on the kinetic energy \(K\) and the dissipation \(\varepsilon\) :

\[\tau_s \sim \frac{K}{\varepsilon}\]

- Characteristic time scale ratio : shear rapidity

\[\frac{\tau_s}{\tau_f} \sim \frac{SK}{\varepsilon} \tag{53}\]

Slow and fast Pressure terms

Pure shear flows

- Momentum equation

\[
\frac{\partial u_i'}{\partial t} + S x_2 \frac{\partial u_i'}{\partial x_1} + S u_i' \delta_{1i} + \frac{\partial}{\partial x_j} (u_i' u_j') = - \frac{\partial p_f'}{\partial x_i} - \frac{\partial p_s'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_k \partial x_k} \tag{54}\]

- Fast pressure component : Poisson problem

\[- \Delta p'_f = 2 S \frac{\partial u_2'}{\partial x_1} \tag{55}\]
Simplification of the budget equations

Reynolds stress $R_{ij}$

- Reynolds stress tensor $R_{ij}$

$$\frac{d}{dt} R_{ij} = -S \begin{pmatrix} 2R_{12} & R_{22} & R_{23} \\ R_{22} & 0 & 0 \\ R_{23} & 0 & 0 \end{pmatrix} + \Pi_{ij} - \varepsilon_{ij} \tag{56}$$

with

$$\Pi_{ij} = \rho S_{ij}, \quad \varepsilon_{ij} = 2\rho \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k} \tag{57}$$

Energetic coupling

Production of kinetic energy

- Produced by the coupling between the mean field and the cross correlation $R_{12}$. The production is anisotropic.
  - Because governed by $R_{12}$ which is identically zero in the isotropic case.
  - It appears only in the $R_{11}$ budget equation.
- There is no pressure contribution in the budget of $K(t)$ (via $\Pi_{ij}$).
  - Indeed, $\Pi_{ii}$ is identically zero, because of the incompressibility condition on $u'$.
  - The pressure role only consists in redistributing the kinetic energy between the components of the Reynolds tensor, without lost or creation.

homogeneous shear case
Measure of the anisotropy

**Anisotropy tensor \( b \)**

\[
b_{ij} = \frac{R_{ij}}{2K} - \frac{\delta_{ij}}{3}.
\] (59)

- \( b_{ij} = 0 \) in the isotropic case
- Traceless tensor: \( b_{ii} = 0 \).

Experimental and numerical observations

**From isotropic state**

- First stage: \( R_{12} \), initially zero, is still small. \( K \) decreases.
- Second stage: The anisotropy and the kinetic energy production strongly increase. At this moment, one observes that \( K \) start to increase.
- Final stage characterized by a universal asymptotic regime of exponential growth of the fluctuating kinetic energy.
  - Invariant in time:
    \[
    \frac{SK}{\varepsilon}, \quad \frac{SR_{12}}{\varepsilon} = 2b_{12} \frac{SK}{\varepsilon}
    \] (60)
  - Equation (58) \( \Rightarrow \)
    \[
    K(t) = K(0)e^{\sigma S}, \quad \sigma = -2b_{12} \left( 1 - 2b_{12} \frac{SK}{\varepsilon} \right) = \text{cste}
    \] (61)

Sheared homogeneous turbulence

**Experimental results: Exponential growth**

![Figure: Time evolution of \( K \) (at left) and \( \sqrt{R_{11}} \) (at right) (extracted from Rohr et al. (1988))](image)

Sheared homogeneous turbulence

**Experimental results**

![Figure: Evolution of the spatial scales \( (L_k : \text{Kolmogorov's scale}, \, \lambda : \text{Taylor's scale}, \, l : \text{integral scale}) \), the solid line denotes the evolutions in the isotropic case. (extracted from Rohr et al. (1988))](image)
Sheared homogeneous turbulence

EDQNM results

![Graph showing EDQNM results](image)

Table: Summary of global quantities obtained in DNS and experiments for shear flows, classified by date.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Kind</th>
<th>Date</th>
<th>( R_{\tau}(0) )</th>
<th>( S^*(0) )</th>
<th>( S^*_{\text{ref}} )</th>
<th>( b_{13} )</th>
<th>( \gamma )</th>
<th>( (St)_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tavoularis et al.</td>
<td>Exp</td>
<td>1981</td>
<td>245</td>
<td>12.5</td>
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<td>(-0.14)</td>
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<td>Shirani et al.</td>
<td>DNS</td>
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<td>0.09</td>
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<td>Lee et al.</td>
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<td>40</td>
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<td>36.2</td>
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<tr>
<td>De Souza et al.</td>
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<tr>
<td>De Souza et al.</td>
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<td>Ferchichi et al.</td>
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<td>253</td>
<td>/</td>
<td>/</td>
<td>/</td>
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<td>/</td>
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<tr>
<td>Brethouwer</td>
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<td>Isaza et al.</td>
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<td>3</td>
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<td>Sukheswala et al.</td>
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<td>Standard deviation</td>
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<td></td>
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<td>10.55</td>
<td>0.027</td>
<td>0.037</td>
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RDT: Simplification of the budget equations

Sheared homogeneous turbulence

Figure: Anisotropy tensor \( b_{ij}(t) \) and shear rapidity \( S_{\eta}(t) \) with \( S = 1/\tau_0^{-1} \).

(a) \( \sigma = 2 \). (b) \( \sigma = 4 \).

Rapid Distortion Theory

Analysis of the fast physical mechanisms

- Non zero mean velocity field. Is the behavior observed are due to rapid (linear in \( u' \)) or slow (non-linear) effects?

\[ \Rightarrow \text{Strong importance for the development of new turbulence models.} \]

RDT: short time response analysis to the instantaneous variation of \( \frac{\Delta t}{\tau_s} \)

- Justified approach if the shear rapidity parameter \( SK/\varepsilon \gg 1 \), i.e.

\[ \tau_f \ll \tau_s \]

- Equations after elimination of slow terms

\[ \frac{\partial u'_i}{\partial t} + A_{jk}x_j \frac{\partial u'_j}{\partial x_k} + A_{ij}u'_j = \frac{\partial p'_j}{\partial x_i} + \nu \frac{\partial^2 u'_j}{\partial x_k \partial x_k} \tag{62} \]

\[ - \Delta p'_j = 2A_{lnm} \frac{\partial u'_n}{\partial x_l} \tag{63} \]
Comparison between Experiment and RDT

Table: Asymptotic behaviors for the case of shear homogeneous flows

<table>
<thead>
<tr>
<th></th>
<th>exp. measures</th>
<th>RDT</th>
<th>RDT without pressure</th>
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<tbody>
<tr>
<td>$\lambda_1$ $(St \gg 1)$</td>
<td>$e^{St}$</td>
<td>$St$</td>
<td>$(St)^2$</td>
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<tr>
<td>$b_{11}$ $(St \gg 1)$</td>
<td>0, 203</td>
<td>$2/3$</td>
<td>$2/3$</td>
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<tr>
<td>$b_{22}$ $(St \gg 1)$</td>
<td>-0.143</td>
<td>$-1/3$</td>
<td>$-1/3$</td>
</tr>
<tr>
<td>$b_{33}$ $(St \gg 1)$</td>
<td>-0.06</td>
<td>$-1/3$</td>
<td>$-1/3$</td>
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<tr>
<td>$b_{12}$ $(St \gg 1)$</td>
<td>-0.15</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- RDT: algebraic growth and not exponential... $\implies$ Exponential growth due to non-linear effects!
- Stabilizing effect of pressure: $t^2 \rightarrow t$.
- Bad prediction of anisotropy behavior at large time, as expected...